Representation of even integers as a sum of squares of primes and powers of two

Shehzad Hathi

UNSW Canberra at ADFA

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Linnik's approximation to Goldbach problem

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  where K is an absolute constant.
- Heath-Brown & Puchta and Pintz & Ruzsa independently proved $K = 7$ under GRH. Recently, Pintz & Ruzsa proved $K = 8$ unconditionally.

Motivation

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- Lagrange's four squares theorem:
  Every natural number can be written as the sum of four squares (of integers).
- Hua's result:
  Every sufficiently large integer $\equiv 5 \mod 24$ can be expressed as the sum of five squares of primes.
- Brüdern and Fouvry:
  Every sufficiently large integer $N \equiv 4 \mod 24$ can be represented as the sum of four squares of almost primes.

Representation as sum of four squares of primes

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- Explicit values of k (powers of two):

$k = 8330$	(Liu & Liu, 2000)
$k = 165$	(Liu & Lü, 2004)
$k = 151$	(Li, 2006)
$k = 46$	(Zhao, 2014)
$k = 45$	(Platt & Trudgian, 2015)
$k = 28$	(H., 202?)

Circle method set-up

Let $R_k(N)$ be the weighted no. of representations of N as four squares of primes and k powers of two:

$$R_k(N) = \sum_{\substack{{p_1}^2+{p_2}^2+{p_3}^2+{p_4}^2+2^{\nu_1}+\cdots+2^{\nu_k}=N \\ 1\le\nu_1,\ldots,\nu_k\le L}} \prod_{j=1}^{4} \log p_j.$$

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As usual, we can express $R_k(N)$ as a sum of integrals over the major arcs ($\mathfrak{M}$) and minor arcs ($\mathfrak{m}$), respectively.

$$R_k(N) = \int_{\mathfrak{M}} T^4(\alpha)G^k(\alpha)e(-\alpha N)d\alpha + \int_{\mathfrak{m}} T^4(\alpha)G^k(\alpha)e(-\alpha N)d\alpha.$$

where

$$T(\alpha) := \sum_p (\log p)e\left(p^2\alpha\right) \quad \text{and} \quad G(\alpha) := \sum_\nu e\left(2^\nu\alpha\right).$$

Defining major arcs and minor arcs

Let

$$L = \frac{\log(N/\log N)}{\log 2}.$$

Set the parameters,

$$P = N^{1/5-\epsilon} \quad \text{and} \quad Q = \frac{N}{PL^{2l}}.$$

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Then:

$$\mathfrak{M} := \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q \\ (a,q)=1}} \mathfrak{M}_{a,q} \quad \text{and} \quad \mathfrak{m} := \left[\frac{1}{Q}, 1+\frac{1}{Q}\right] \setminus \mathfrak{M},$$

where

$$\mathfrak{M}_{a,q} := \left\{\alpha \,:\, \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}\right\}.$$

Estimate over major arcs

Lemma (Zhao)

Let

$$\mathfrak{J}(h) = \int_{-\infty}^{\infty} \left(\int_{\sqrt{1/4-\eta}}^{\sqrt{1/4+\eta}} e\left(x^2\beta\right) dx \right)^4 e(-h\beta) d\beta.$$

Then

$$\int_{\mathfrak{M}} T^4(\alpha) G^k(\alpha) e(-\alpha N) d\alpha \geq 0.9 \times 8\mathfrak{J}(1) N L^k + O\left(N L^{k-1}\right).$$

Estimate over minor arcs

Lemma (Zhao)

We have

$$\int_{\mathfrak{m}} \left| T(\alpha)^4 G(\alpha)^{2l} \right| d\alpha \leq 8(15+\epsilon) c_{0,l} \left(1 + O(\eta)\right) \mathfrak{J}(0) N L^{2l} + O\left(N L^{2l-1}\right).$$

Why do we want to estimate $c_{0,l}$?

For $N \equiv 4 \mod 8$, we have

$$R_{k'}(N) \geq 8NL^{k'}\left(0.9\mathfrak{J}(1) - {\lambda_0}^{k'-2l}(15+\epsilon)c_{0,l}(1+O(\eta))\mathfrak{J}(0)\right).$$

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We would like to prove $R_{k'}(N) > 0$ for sufficiently large $N \equiv 4 \mod 8$. Then $R_k(N) > 0$ for $k = k' + 2$ and sufficiently large even N.

Computation of $c_{0,l}$

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We can split $c_{1,l}$ into exact and inexact parts:

$$c_{1,l} \leq \sum_{\substack{p \mid d \Rightarrow p > 5 \\ d < (2^M - 1)/3 \\ \beta_l(3d) < M}} \frac{\mu^2(d)}{c(d)} \left(\frac{1}{\beta_l(3d)} - \frac{1}{M} \right) + \text{inexact part}$$

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Here,

$$\beta_l(3d) := \frac{\rho^{2l}(3d)}{N_l(3d)}$$

where

$$N_l(3d) := \sum_{\substack{3d|\sum_{1\leq j\leq l}(2^{u_j}-2^{v_j}) \\ 1\leq u_j,v_j\leq \rho(3d)}} 1.$$

Relevant inequalities

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Also,

$$\beta_l(d) \geq \beta_{l-1}(d)$$

Table: $\beta_7(3d)$ for large primes

p	$\beta_7(3p)$	Time (in s)
22366891	3089168.27	15238
25781083	15652237.24	19914
164511353	56626483.49	13332
616318177	28269951.69	6728

Choice of l

In Zhao's work, l is chosen to be 7. Zhao's estimate is $c_{0,7} < 2.07$*. Using our computations, we obtain $c_{0,7} < 0.78$. But here, we choose $l = 2$ ($c_{0,2} < 0.803$) due to the following table.

Table: Powers of two for different values of l

l	k
2	28
3	29
4	31
5	33
6	35
7	37
8	39

Applications

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- Liu (2014) obtained a refinement in the problem of representing N as

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- Hu and Liu (2015) have considered the problem of simultaneous representations:

$$N_1 = {p_1}^2 + {p_2}^2 + {p_3}^2 + {p_4}^2 + 2^{\nu_1} + \cdots + 2^{\nu_k},$$
$$N_2 = {q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2 + 2^{\nu_1} + \cdots + 2^{\nu_k}.$$

  They showed that $k = 142$ is acceptable.