Investigation - Euler Characteristic

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July 26, 2016

A **polyhedron** is a 3D solid with straight edges and faces.

1. Is a sphere a polyhedron? Justify.

In a polyhedron, if two faces meet, they have a common edge between them and if two edges meet, they share a vertex. **Euler characteristic** or **Euler number** is a specific relation between the number of vertices (V), number of edges (E) and number of faces (F) for a polyhedron. (*What are the dimensions of a vertex, an edge and a face? Is there any connection between 2D polygons and 3D polyhedrons?*)

Count V, E, F for regular convex polyhedra, also called **platonic solids** (see Figure 2). (*What are regular convex polyhedra?*)

2. Fill V,E,F for Platonic Solids

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
V		8		20	12
E		12		30	30
F		6			20

If I tell you that there is a specific linear relation between V, E, and F that all of these platonic solids satisfy, can you solve three simultaneous equations and find out the relation? Assume the identity to be

$$xV + yE + zF = k,$$

where x,y,z, and k are real numbers. Dividing the equation by x, we can rewrite it as

$$V + aE + bF = c.$$

3. Find the values of a,b, and c and check whether all platonic solids satisfy this identity.

In fact, all convex polyhedra should satisfy this identity (*Why?*), called **Euler's polyhedron formula**. In general, Euler characteristic,

$$\chi = V - E + F,$$

is an important quantity that describes the shape of a 3D object regardless of the way it is bent. For instance, we can prove that there are only five platonic solids using the value of χ.

4. For convex polyhedra, the value of $\chi =$

Let us apply what we learnt on a **soccer ball**. A soccer ball is made of pentagons and hexagons stitched together and can be considered a convex polyhedron (see Figure 1). (*Convince yourself by looking at one!*) That means its Euler characteristic has to be the constant value that you wrote in question 4. Now, in this polyhedron, at every vertex three shapes (hexagons and pentagons) meet. Let the number of pentagons and hexagons in the ball be P and H respectively. Then the number of vertices, V, in terms of P and H is

$$V = (5P + 6H)/3$$

because each pentagon has five vertices and each hexagon has six vertices and each vertex is counted once in three different shapes.

5. In a similar manner, write equations for E and F in terms of P and H.

6. Substitute these expressions in the Euler's polyhedron formula and simplify.

7. Is there any restriction on the number of pentagons and the number of hexagons in a soccer ball? Conclude from question 6.

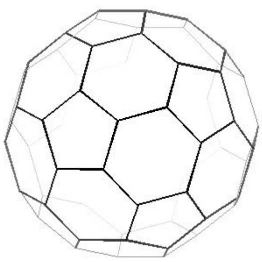
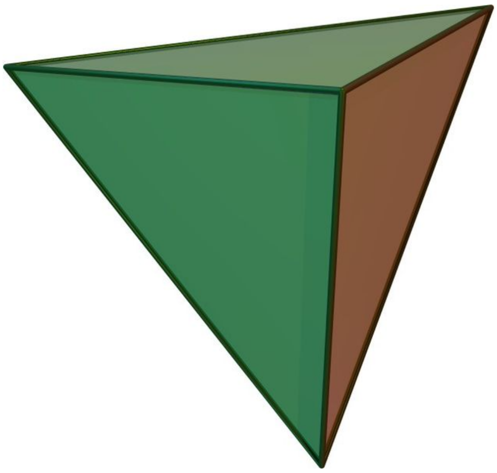
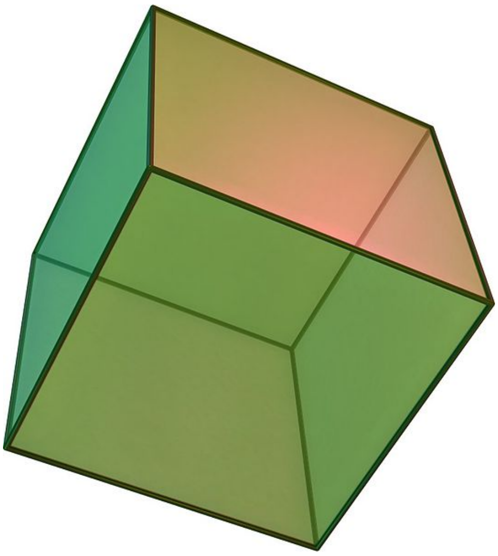


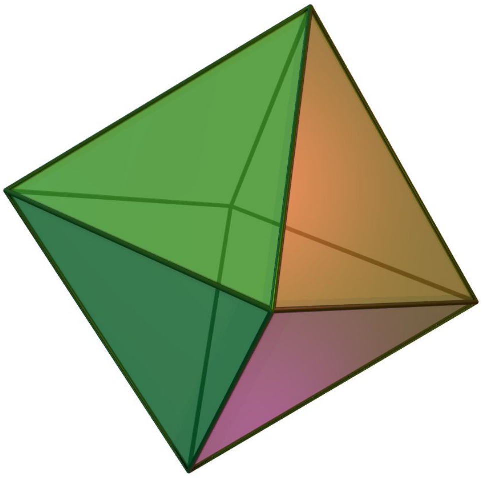
Figure 1: Soccer Ball (Source: MathWorks)



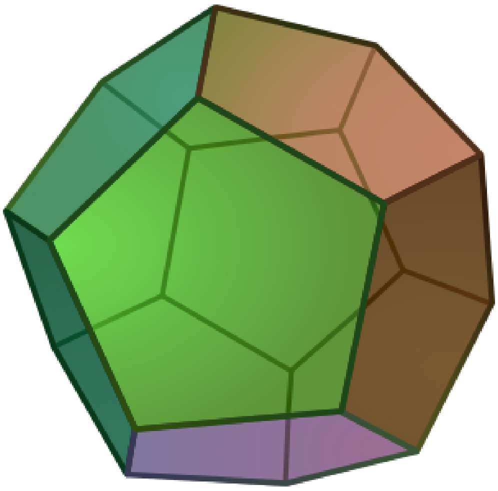
(a) Tetrahedron



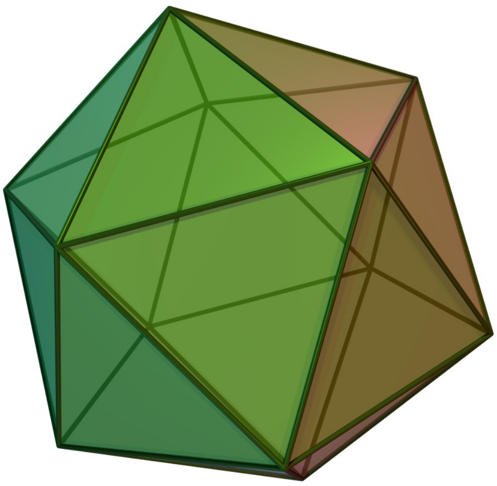
(b) Cube



(c) Octahedron



(d) Dodecahedron



(e) Icosahedron

Figure 2: Platonic Solids (Source: Wikipedia)