Biases, oscillations, and first sign changes

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Arithmetic functions

Arithmetic functions are complex-valued functions defined on positive integers. For instance, we have the characteristic function of primes:

$$\chi_{\text{prime}}(n) = \begin{cases} 1 & \text{if n is prime} \\ 0 & \text{otherwise} \end{cases}$$

Its summatory function is the prime counting function that is defined for all positive reals:

$$\pi(x) = \sum_{n \le x} \chi_{\text{prime}}(n) = \sum_{\substack{p \le x \\ p \text{ prime}}} 1.$$

Prime number theorem

The asymptotic behaviour of $\pi(x)$ is given by

$$\pi(x) \sim \mathrm{Li}(x) = \int_2^x \frac{dt}{\log t},$$

which is referred to as the prime number theorem (PNT). It was first proven by Hadamard and de la Vallée Poussin independently in 1896.

Error term in PNT

In fact, de la Vallée Poussin also proved the relative error in this estimation, that is,

$$\left| \frac{\pi(x) - \mathrm{Li}(x)}{\mathrm{Li}(x)} \right| < e^{-c\sqrt{\log x}}$$

for some positive constant c and for all sufficiently large x.

Under the Riemann Hypothesis (RH), it can be shown that

$$\left| \frac{\pi(x) - \mathrm{Li}(x)}{\mathrm{Li}(x)} \right| < x^{-(1/2)+\epsilon}$$

as $x \to \infty$ and for all $\epsilon > 0$.

Sign of $\pi(x) - \mathrm{Li}(x)$

Possibilities:

- It **maintains the same sign** for all x. As an example, one could look at the other formulation of PNT

$$\pi(x) \sim \frac{x}{\log x}.$$

  In this case, we know that $\pi(x) - (x/\log x)$ is always positive for $x \geq 2$.

- $\pi(x)$ **oscillates** about $\mathrm{Li}(x)$ with both signs featuring "approximately equally", or
- It changes sign "quite often" but exhibits a **bias** towards a certain sign.

Skewes' number

In this particular case, $\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often [Lit14] but exhibits a very strong bias towards the negative sign.

The density of numbers where $\pi(x) - \mathrm{Li}(x) < 0$ is very close to 1 but not exactly 1 ([Win41] and [RS94]). If the bias towards a certain sign is quite strong initially, then it also becomes interesting to ask the question about the **first sign change**. The value of x where a sign change occurs (or the smallest such number) from negative to positive is called *Skewes' number*.

While we do not have an explicit value for a Skewes' number, there are upper bounds for the first Skewes' number [STD15]. It is also known that $\pi(x) < \mathrm{Li}(x)$ upto 10^{19} [Büt18].

Möbius function

The Möbius function, $\mu(\cdot)$, is a standard arithmetic function defined in the following manner:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdot p_2 \cdots p_k \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer and $p_1, p_2, \ldots, p_k$ are distinct primes.

The Möbius function is related to the zeta-function by the following property:

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$

for $\Re(s) \geq 1$.

Mertens' conjecture

As before, we consider the summatory function

$$M(x) := \sum_{n \leq x} \mu(n).$$

The Mertens' conjecture is the claim that

$$|M(x)| < C\sqrt{x}$$

for $x > 1$ and some $C \geq 1$. In fact, $M(x) = O(x^{\theta})$ implies that the Riemann zeta function, $\zeta(s)$, has no zeros in the half-plane $\Re(s) > \theta$.

Implications

Let a zero on the critical line be $\rho_\nu := \beta_\nu + i\gamma_\nu$.

Apart from the truth of RH, the Mertens' conjecture also implies

- simplicity of the zeros, and
- γ_ν's linearly dependent [Ing42].

Although the numerical evidence supporting the Mertens' conjecture is substantial (there are no counter-examples until 10^{16} [Hur18]), the last implication suggests that the conjecture is too good to be true since there is no reason why such a linear dependence should exist.

N-independence

Ingham's method, in general, gives us arbitrarily large oscillations in the value of $M(x)/\sqrt{x}$ if we assume linear independence of γ_ν's, i.e.

$$\sum_{\nu} c_\nu \gamma_\nu = 0, \quad c_\nu \in \mathbb{Z}$$

does not have any non-trivial solution. Since establishing true linear independence of γ_ν's is not feasible, one could ask if a weaker form of linear independence gives us large but finite oscillations.

We say that the first m zeros (on the critical line) are *N-independent* if

$$\sum_{i=1}^{m} c_i \gamma_i = 0, \quad c_i \in \mathbb{Z},$$

with $|c_i| \leq N$, does not have any non-trivial solution.

Disproof and counterexamples

Odlyzko and te Riele [OR85] succeeded in proving that $\limsup_{x\to\infty} M(x)/\sqrt{x} > 1.06$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} < -1.009$, thereby disproving the conjecture (for $C = 1$).

However, the method of the disproof was ineffective as it did not yield a counterexample or even an X such that

$$\max_{1\le x\le X} \frac{|M(x)|}{\sqrt{x}} > 1.$$

Subsequently, Pintz [Pin87] gave an effective disproof with $X = e^{3.21\cdot 10^{64}}$. Since then, the upper bound has been improved, more recently, by Saouter and te Riele [SR14] to $e^{1.004\cdot 10^{33}}$.

Growth of the Mertens' function

While the exact growth of $M(x)$ is not known, all the conjectures point to a smaller counterexample than the one proved in [SR14]. For instance, Kotnik and te Riele [KR06] conjectured that

$$\frac{M(x)}{\sqrt{x}} = \Omega_{\pm}(\sqrt{\log\log\log x}),$$

which suggests that the first counterexample might be at $\simeq e^{5\cdot 10^{23}}$. Kaczorowski's work [Kac07] suggests that the first counterexample could be at $\simeq 10^{703}$, as noted in [SR14]. Both of these values for X are well below what has been proven so far ($e^{1.004\cdot 10^{33}}$).

Future work

- Improve upon Saouter and te Riele's bound using more recent results about the modulus and the zeros of the zeta-function.
- Refine aspects of Ingham's method with respect to N-independence.
- Analyse biases, oscillations and sign changes in other settings, for instance, prime number races. Consider arithmetic functions $f_a(\cdot)$ and $f_b(\cdot)$ that take non-zero values when $p \equiv a \bmod q$ and $p \equiv b \bmod q$ respectively, where p is a prime. We can ask similar questions (as in the case of $\pi(x) - \mathrm{Li}(x)$) about $\sum_{n \leq x}(f_a(x) - f_b(x))$.

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